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Group analysis for unsteady axisymmetric incompressible viscous flow (kinematic approach)

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Abstract. Group properties are investigated for unsteady axisymmetric incompressible viscous flow by means of the kinematic approach of Pillow and Paull. The full symmetry group and Lie algebra for the original system of three partial differential equations is derived and is shown to be infinite dimensional. Further group reductions are possible and some solutions are constructed.

1. Introduction

Three basic kinematic conservation principles suffice to describe unsteady axisymmetric incompressible viscous flow (Pillow 1970). They concern the conservation of volume, ring (axial half-plane) circulation (with volume density l) and kinematic swirl angular momentum (with volume density T). The dynamic role of the pressure field is relegated to the equations of motion. Following Pillow and Paull (1985), the system of governing equations is cast in spherical polar coordinates (r, θ, ϕ). Thus, away from the axis of symmetry, we have

$$r^4(1-\mu^2)l+(1-\mu^2)\psi_{\mu\mu}+r^2\psi_{rr}=0 \tag{1.1}$$

$$r^5(1-\mu^2)^2l_t+r^3(1-\mu^2)^2(\psi_r l_\mu-\psi_\mu l_r)-\nu r^3(1-\mu^2)^2[(1-\mu^2)l_{\mu\mu}-4\mu l_\mu+r^2 l_{rr}+4rl_r]$$
$$-2(1-\mu^2)TT_\mu-2r\mu TT_r=0 \tag{1.2}$$

$$r^2T_t+\psi_r T_\mu-\psi_\mu T_r-\nu[(1-\mu^2)T_{\mu\mu}+r^2T_{rr}]=0. \tag{1.3}$$

Here ψ and ν are the stream function and the (constant) kinematic viscosity, respectively, and $\mu=\cos\theta$. To simplify the notation, we make the following substitutions in (1.1)-(1.3): x instead of r, y instead of μ, u instead of l, v instead of ψ and w instead of T.

Then we have

$$x^4(1-y^2)u+(1-y^2)v_{yy}+x^2v_{xx}=0 \tag{1.4}$$

$$x^4(1-y^2)u_t+x^3(1-y^2)^2(v_xu_y-v_yu_x)-\nu x^3(1-y^2)^2[(1-y^2)u_{yy}-4yu_y+x^2u_{xx}+4xu_x]$$
$$-2(1-y^2)ww_y-2xyww_x=0 \tag{1.5}$$

$$x^2w_t+v_xw_y-v_yw_x-\nu[(1-y^2)w_{yy}+x^2w_{xx}]=0. \tag{1.6}$$

In this paper, the group properties for the system (1.4)-(1.6) are developed. Two subgroups of the full group are used to generate exact solutions, further group reduction being possible.

2. Full Lie group and algebra

The mathematical foundations for the determination of the full group for a system of differential equations can be found in Ames (1972) and Bluman and Cole (1974) and the general theory is found in Ovsiannikov (1982). In the spirit of Lie we desire to find infinitesimal transformations of the form:

$$\begin{aligned}
t' &= t+\varepsilon T(t, x, y, u, v, w)+O(\varepsilon^2)\\
x' &= x+\varepsilon X(t, x, y, u, v, w)+O(\varepsilon^2)\\
y' &= y+\varepsilon Y(t, x, y, u, v, w)+O(\varepsilon^2)\\
u' &= u+\varepsilon U(t, x, y, u, v, w)+O(\varepsilon^2)\\
v' &= v+\varepsilon V(t, x, y, u, v, w)+O(\varepsilon^2)\\
w' &= w+\varepsilon W(t, x, y, u, v, w)+O(\varepsilon^2)
\end{aligned} \tag{2.1}$$

which leave (1.4)-(1.6) invariant. System (2.1) leaves (1.4)-(1.6) invariant if and only if (u', v', w') is a solution of (1.4')-(1.6') whenever (u, v, w) is a solution to (1.4)-(1.6). By (1.4')-(1.6') we mean the same equations in the primed variables. By extensive analysis it is found that the full Lie group leaving (1.4)-(1.6) invariant is given by (2.1) with

$$T=2\alpha t+\beta \tag{2.2}$$

$$X=\alpha x+h(t)y \tag{2.3}$$

$$Y=\frac{1-y^2}{x}h(t) \tag{2.4}$$

$$U=-3\alpha u \tag{2.5}$$

$$V=\alpha v+\tfrac{1}{2}x^2(1-y^2)h'(t)+f(t) \tag{2.6}$$

$$W=0 \tag{2.7}$$

where α and β are two arbitrary parameters and $h(t)$ and $f(t)$ are arbitrary, sufficiently smooth functions of t. With X_i $(i=1,2)$ representing the generators associated with the parameters α and β, respectively, it follows that

$$X_1=2t\partial_t+x\partial_x-3u\partial_u+v\partial_v \tag{2.8}$$

$$X_2=\partial_t. \tag{2.9}$$

Also, from the arbitrary functions in (2.3)-(2.6) infinitely many operators of the following forms are obtained:

$$X_3(h)=h(t)y\partial_x+\frac{1-y^2}{x}h(t)\partial_y+\frac{x^2}{2}(1-y^2)h'(t)\partial_v \tag{2.10}$$

$$X_4(f)=f(t)\partial_v. \tag{2.11}$$

Table 1.

	X_1	X_2	$X_3(h)$	$X_4(f)$
X_1	0	$-2X_1$	$X_3(2th'-h)$	$X_4(2tf'-f)$
X_2	$2X_1$	0	$X_3(h')$	$X_4(f')$
$X_3(h)$	$-X_3(2th'-h)$	$-X_3(h')$	0	0
$X_4(f)$	$-X_4(2tf'-f)$	$-X_4(f')$	0	0

The commutator table of the Lie algebra for the system (1.4)-(1.6) is given in table 1, where the entry in the ith row and jth column is the commutator of X_i, X_j, i.e.

$$|X_i, X_j| = X_iX_j - X_jX_i. \tag{2.12}$$

3. The subgroup generated by the parameter α

We consider the subgroup of the full Lie group (2.2)-(2.7) with $\alpha = 1$ and $\beta = h(t) = f(t) = 0$. This subgroup has the associated operator

$$X_1 = 2t\partial_t + x\partial_x - 3u\partial_u + v\partial_v. \tag{3.1}$$

The first-order equation for the invariants $X_1 I = 0$ has the characteristics

$$\xi = t/x^2 \qquad \eta = y \tag{3.2}$$

$$u = \tilde{u}(\xi, \eta)/x^3, \qquad v = x\tilde{v}(\xi, \eta) \quad w = \tilde{w}(\xi, \eta). \tag{3.3}$$

Under (3.2) and (3.3) the system (1.4)-(1.6) becomes

$$(1-\eta^2)\tilde{u} + (1-\eta^2)\tilde{v}_{\eta\eta} + 2\xi\tilde{v}_\xi + 4\xi^2\tilde{v}_{\xi\xi} = 0 \tag{3.4}$$

$$\begin{aligned}(1-\eta^2)^2\tilde{u}_\xi &+ (1-\eta^2)^2(\tilde{v}\tilde{u}_\eta + 3\tilde{u}\tilde{v}_\eta - 2\xi\tilde{v}_\xi\tilde{u}_\eta + 2\xi\tilde{u}_\xi\tilde{v}_\eta)\\ &- \nu(1-\eta^2)^2[(1-\eta^2)\tilde{u}_{\eta\eta} - 4\eta\tilde{u}_\eta + 10\xi\tilde{u}_\xi + 4\xi^2\tilde{u}_{\xi\xi}]\\ &- 2(1-\eta^2)\tilde{w}\tilde{w}_\eta + 4\xi\eta\tilde{w}\tilde{w}_\xi = 0\end{aligned} \tag{3.5}$$

$$\tilde{w}_\xi + \tilde{v}\tilde{w}_\eta - 2\xi\tilde{v}_\xi\tilde{w}_\eta + 2\xi\tilde{v}_\eta\tilde{w}_\xi - \nu[(1-\eta^2)\tilde{w}_{\eta\eta} + 6\xi\tilde{w}_\xi + 4\xi^2\tilde{w}_{\xi\xi}] = 0. \tag{3.6}$$

Further group reduction is possible. By lengthy calculations it is found that the full Lie group leaving (3.4)-(3.6) invariant is given by

$$\begin{aligned}\xi' &= \xi + \varepsilon(2c_1\xi\eta\sqrt{\xi}) + O(\varepsilon^2)\\ \eta' &= \eta + \varepsilon[-c_1(1-\eta^2)\sqrt{\xi}] + O(\varepsilon^2)\\ \tilde{u}' &= \tilde{u} + \varepsilon(-3c_1\tilde{u}\eta\sqrt{\xi}) + O(\varepsilon^2)\\ \tilde{v}' &= \tilde{v} + \varepsilon\left[\left(c_1\eta\tilde{v} - c_1\frac{1-\eta^2}{4\xi} + c_2\right)\sqrt{\xi}\right] + O(\varepsilon^2)\\ \tilde{w}' &= \tilde{w} + O(\varepsilon^2)\end{aligned} \tag{3.7}$$

where c_1 and c_2 are two parameters. It corresponds to a finite-dimensional Lie algebra L_2. The equations for the invariant surface are obtained from (for $c_1 \neq 0$)

$$\zeta = (1-\eta^2)/\xi \tag{3.8}$$

$$\tilde{u} = \xi^{-3/2} F(\zeta) \qquad \tilde{v} = \eta\left(\frac{1-\eta^2}{4\xi} - \frac{c_2}{c_1}\right) + \xi^{1/2} G(\zeta)$$
$$\tilde{w} = H(\zeta) \tag{3.9}$$

where $F(\zeta)$, $G(\zeta)$, $H(\zeta)$ satisfy the ordinary differential equations

$$F + 4G'' = 0 \tag{3.10}$$

$$8\nu\zeta F'' + (3\zeta + 16\nu - 4c_2/c_1)F' + 3F = 0 \tag{3.11}$$

$$8\nu\zeta H'' + (3\zeta - 4c_2/c_1)H' = 0. \tag{3.12}$$

If we assume $c_2 = 2\nu c_1$, the general solution of (3.12) is given by

$$H(\zeta) = -\tfrac{8}{3}\nu A_1(\zeta + \tfrac{8}{3}\nu)\exp(-3\zeta/8\nu) + A_2 \tag{3.13}$$

with A_1, A_2 arbitrary constants. We also obtain the general solution of (3.11):

$$F(\zeta) = A_3 \exp(-3\zeta/8\nu) + (A_6/8\nu)\left[\int \zeta^{-1}\exp\left(\frac{+3\zeta}{8\nu}\right) \mathrm{d}\zeta\right]\exp(-3\zeta/8\nu) \tag{3.14}$$

with A_3, A_6 arbitrary constants. Then from (3.10) we have

$$G(\zeta) = -\frac{16}{9}\nu^2 A_3 \exp\left(-\frac{3\zeta}{8\nu}\right) + A_4\zeta + A_5$$
$$-\frac{A_6}{32\nu}\int\left\{\int \exp\left(-\frac{3\zeta}{8\nu}\right)\left[\int \zeta^{-1}\exp\left(\frac{3\zeta}{8\nu}\right)\mathrm{d}\zeta\right]\mathrm{d}\zeta\right\}\mathrm{d}\zeta \tag{3.15}$$

with A_4, A_5 arbitrary constants. Assuming $A_6 = 0$, the solution of the system (1.4)-(1.6) resulting from (3.13)-(3.15) is

$$u(t,x,y) = \frac{A_3}{t\sqrt{t}}\exp\left(-\frac{3(1-y^2)x^2}{8\nu t}\right) \tag{3.16}$$

$$v(t,x,y) = -2\nu xy + \frac{(1-y^2)x^3 y}{4t} - \tfrac{16}{9}\nu^2 A_3\sqrt{t}\exp\left(-\frac{3(1-y^2)x^2}{8\nu t}\right)$$
$$+ A_4\frac{(1-y^2)x^2}{\sqrt{t}} + A_5\sqrt{t} \tag{3.17}$$

$$w(t,x,y) = -\tfrac{8}{3}\nu A_1\left(\frac{(1-y^2)x^2}{t} + \tfrac{8}{3}\nu\right)\exp\left(-\frac{3(1-y^2)x^2}{8\nu t}\right) + A_2. \tag{3.18}$$

4. The subgroup with $\beta = 1$ and $\alpha = 0$

Here we consider the subgroup of the full Lie group (2.2)-(2.7) with the associated operator

$$\bar{Q} = \partial_t + h(t)y\partial_x + \frac{1-y^2}{x}h(t)\partial_y + [\tfrac{1}{2}x^2(1-y^2)h'(t) + f(t)]\partial_v. \tag{4.1}$$

It gives rise to the characteristic variables

$$\xi_1 = x(1-y^2)^{1/2} \qquad \xi_2 = xy - \bar{h}(t) \tag{4.2}$$

$$u = \bar{u}(\xi_1, \xi_2) \qquad v = \tfrac{1}{2}x^2(1-y^2)h(t) + \bar{f}(t) + \bar{v}(\xi_1, \xi_2)$$
$$w = \bar{w}(\xi_1, \xi_2) \tag{4.3}$$

where $\bar{h}'(t) = h(t)$ and $\bar{f}'(t) = f(t)$. Under (4.2) and (4.3) the system (1.4)-(1.6) becomes

$$\xi_1^3\bar{u} - \bar{v}_{\xi_1} + \xi_1\bar{v}_{\xi_2\xi_2} = 0 \tag{4.4}$$

$$\xi_1^3(\bar{u}_{\xi_2}\bar{v}_{\xi_1} - \bar{u}_{\xi_1}\bar{v}_{\xi_2}) - 3\nu\xi_1^3\bar{u}_{\xi_1} - \nu\xi_1^4(\bar{u}_{\xi_1\xi_1} + \bar{u}_{\xi_2\xi_2}) - 2\bar{w}\bar{w}_{\xi_2} = 0 \tag{4.5}$$

$$\bar{w}_{\xi_2}\bar{v}_{\xi_1} - \bar{w}_{\xi_1}\bar{v}_{\xi_2} - \nu\xi_1(\bar{w}_{\xi_1\xi_1} + \bar{w}_{\xi_2\xi_2}) + \nu\bar{w}_{\xi_1} = 0. \tag{4.6}$$

It is possible to calculate the full Lie group leaving (4.4)-(4.6) invariant. It is given by

$$\begin{aligned}
\xi_1' &= \xi_1 + \varepsilon(a_1\xi_1) + \mathrm{O}(\varepsilon^2)\\
\xi_2' &= \xi_2 + \varepsilon(a_1\xi_2 + a_2) + \mathrm{O}(\varepsilon^2)\\
\bar{u}' &= \bar{u} + \varepsilon(-3a_1\bar{u}) + \mathrm{O}(\varepsilon^2)\\
\bar{v}' &= \bar{v} + \varepsilon(a_1\bar{v} + a_3) + \mathrm{O}(\varepsilon^2)\\
\bar{w}' &= \bar{w} + \mathrm{O}(\varepsilon^2)
\end{aligned} \tag{4.7}$$

where a_1, a_2, a_3 are three parameters. Then the corresponding Lie algebra is a finite-dimensional Lie algebra L_3. The equations for the invariant surface are found from (for $a_1 \neq 0$)

$$\tau = (a_1\xi_2 + a_2)/a_1\xi_1 \tag{4.8}$$

$$\bar{u} = \Gamma(\tau)/\xi_1^3 \qquad \bar{v} = -(a_3/a_1) + \xi_1\Lambda(\tau) \qquad \bar{w} = \Omega(\tau) \tag{4.9}$$

where $\Gamma(\tau)$, $\Lambda(\tau)$ and $\Omega(\tau)$ satisfy the ordinary differential equations

$$\Gamma - \Lambda + \tau\Lambda' + \Lambda'' = 0 \tag{4.10}$$

$$\Gamma'\Lambda + 3\Gamma\Lambda' - 3\nu\Gamma - 5\nu\tau\Gamma' - \nu(1+\tau^2)\Gamma'' - 2\Omega\Omega' = 0 \tag{4.11}$$

$$\Omega'\Lambda - 3\nu\tau\Omega' - \nu(1+\tau^2)\Omega'' = 0. \tag{4.12}$$

A solution of system (4.10)-(4.12) is

$$\Gamma(\tau) = B_1 \qquad \Lambda(\tau) = \nu\tau + B_1 \qquad \Omega(\tau) = B_2 \tag{4.13}$$

where B_1 and B_2 are arbitrary constants. The solution of system (1.4)-(1.6) resulting from (4.13) is

$$u = B_1x^{-3}(1-y^2)^{-3/2} \tag{4.14}$$

$$v = \frac{\nu c_2 - c_3}{c_1} + \nu xy - \nu\bar{h}(t) + \bar{f}(t) + \tfrac{1}{2}x^2(1-y^2)h(t) + B_1x(1-y^2)^{1/2} \tag{4.15}$$

$$w = B_2. \tag{4.16}$$

5. The swirl-free problem

In swirl-free viscous flows (Pillow 1970, Pillow and Paull 1985) the governing equations (1.4)-(1.6) reduce to

$$x^4(1-y^2)u+(1-y^2)v_{yy}+x^2v_{xx}=0 \tag{5.1}$$

$$x^2u_t+v_xu_y-v_yu_x-\nu[(1-y^2)u_{yy}-4yu_y+x^2u_{xx}+4xu_x]=0 \tag{5.2}$$

since the swirl circulation w is everywhere zero. The full Lie group for the system (5.1) and (5.2) is given by (2.2)-(2.6). Solutions for the system (5.1) and (5.2) are (3.16) and (3.17), and (4.14) and (4.15).

6. Comment

In the following, we return to the original notation of § 1. Using cylindrical polar coordinates (x, σ, ϕ) with x measured along the axis of symmetry, σ perpendicular to it and ϕ the azimuthal angle (so the that the tangent vectors to the coordinate lines $(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\phi})$ form a right-handed orthonormal triad with natural basis vectors $(\boldsymbol{\chi}, \boldsymbol{\sigma}, \sigma\boldsymbol{\phi})$), the velocity $\boldsymbol{q}$ and the vorticity $\boldsymbol{\omega}$ can be written in the form

$$\boldsymbol{q}=\frac{\psi_\sigma}{\sigma}\boldsymbol{x}-\frac{\psi_x}{\sigma}\boldsymbol{\sigma}+\frac{T}{\sigma}\boldsymbol{\phi} \tag{6.1}$$

$$\boldsymbol{\omega}=\frac{T_\sigma}{\sigma}\boldsymbol{x}-\frac{T_x}{\sigma}\boldsymbol{\sigma}+\sigma l\boldsymbol{\phi} \tag{6.2}$$

where ψ, T and l satisfy (1.1)-(1.3).

In § 3, by applying the first group reduction, we obtained

$$l=\hat{u}(\xi,\eta)/r^3 \qquad \psi-r\hat{v}(\xi,\eta) \qquad T=\hat{w}(\xi,\eta) \tag{6.3}$$

with

$$\xi=t/r^2 \qquad \eta=\mu \tag{6.4}$$

where $r^2=x^2+\sigma^2$ and $\mu=x/r$.

Solutions can be generated by solving the system (3.4)-(3.6) for $\hat{u}$, $\hat{v}$ and $\hat{w}$. With a further group reduction, we found

$$l=F(\zeta)/t\sqrt{t} \qquad \psi=x(\sigma^2/4t-2\nu)+\sqrt{t}G(\zeta) \qquad T=H(\zeta) \tag{6.5}$$

with

$$\zeta=\sigma^2/t \tag{6.6}$$

where F, G and H are given by (3.14), (3.15) and (3.13), respectively.

It is a remarkable fact that (3.13)-(3.15) represent the general solution of the 'group reduced' system. Assuming $A_6=0$, the velocity $\boldsymbol{q}$ and the vorticity $\boldsymbol{\omega}$, away from the axis of symmetry, result in

$$\boldsymbol{q}=\left[\frac{x}{2t}+\frac{4\nu}{3\sqrt{t}}A_3\exp\left(-\frac{3\sigma^2}{8\nu t}\right)+\frac{2A_4}{\sqrt{t}}\right]\boldsymbol{x}+\left(\frac{2\nu}{\sigma}-\frac{\sigma}{4t}\right)\boldsymbol{\sigma}$$
$$+\left[-\tfrac{8}{3}\nu A_1\left(\frac{\sigma}{t}+\frac{8\nu}{3\sigma}\right)\exp\left(-\frac{3\sigma^2}{8\nu t}\right)+\frac{A_2}{\sigma}\right]\boldsymbol{\phi} \tag{6.7}$$

$$\boldsymbol{\omega}=\frac{2\sigma^2}{t^2}A_1\exp\left(-\frac{3\sigma^2}{8\nu t}\right)\boldsymbol{x}+\frac{\sigma}{t\sqrt{t}}A_3\exp\left(-\frac{3\sigma^2}{8\nu t}\right)\boldsymbol{\phi} \tag{6.8}$$

with A_i ($i = 1, 2, 3, 4$) arbitrary constants. If we confine the fluid within an impenetrable cone given by

$$\sigma = \sqrt{8\nu t} \tag{6.9}$$

both the σ component and the swirling component of (6.7) satisfy the no-slip condition if

$$A_2 = \tfrac{256}{9}\nu^2 \, e^{-3} A_1 . \tag{6.10}$$

Such a cone generalises, with the introduction of time, the conical boundary for the steady state discussed in Yih *et al* (1982).

In § 4, we obtained

$$l = \bar{u}(\xi_1, \xi_2) \qquad \psi = \tfrac{1}{2}r^2(1-\mu^2)h(t) + \bar{f}(t) + \bar{v}(\xi_1, \xi_2) \qquad T = \bar{w}(\xi_1, \xi_2) \tag{6.11}$$

with

$$\xi_1 = r(1-\mu^2)^{1/2} = \sigma \qquad \xi_2 = r\mu - \bar{h}(t) = x - \bar{h}(t). \tag{6.12}$$

Here $\bar{h}'(t) = h(t)$, and $h(t)$, $\bar{f}(t)$ are arbitrary functions of time. Again, solutions can be generated solving the system (4.4)-(4.6) in $\bar{u}$, $\bar{v}$ and $\bar{w}$. Applying a group reduction, it turned out that

$$l = \Gamma(\tau)/\sigma^3 \qquad \psi = \tfrac{1}{2}\sigma^2 h(t) + \bar{f}(t) - a_3/a_1 + \sigma\Lambda(\tau) \qquad T = \Omega(\tau) \tag{6.13}$$

with

$$\tau = \frac{a_1[x - \bar{h}(t)] + a_2}{a_1\sigma} \tag{6.14}$$

where Γ, Λ and Ω satisfy the system (4.10)-(4.12). Numerical results can be easily obtained. The particular exact solution found generates the following velocity and vorticity:

$$\boldsymbol{q} = [\bar{h}'(t) + B_1/\sigma]\boldsymbol{x} - (\nu/\sigma)\boldsymbol{\sigma} + (B_2/\sigma)\boldsymbol{\phi} \tag{6.15}$$

$$\boldsymbol{\omega} = (B_1/\sigma^2)\boldsymbol{\phi} \tag{6.16}$$

This solution generalises, with the introduction of $\bar{h}'(t)$, one discussed in Pillow (1970) for the steady state.

References

Ames W F 1972 *Nonlinear Partial Differential Equations in Engineering* vol 2 (New York: Academic) ch 2
Bluman G W and Cole J D 1974 *Similarity Methods for Differential Equations* (Berlin: Springer)
Ovsiannikov L V 1982 *Group Analysis of Differential Equations* (New York: Academic)
Pillow A F 1970 *Bull. Austr. Math. Soc.* **2** 145-64
Pillow A F and Paull R 1985 *J. Fluid Mech.* **155** 327-41
Yih C-S, Wu F, Garg A K and Leibovich S 1982 *Phys. Fluids* **25** 2147-58